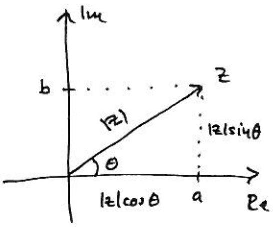
Föreläsning 12

①

Komplexa tal på polär form:



$$z = a+bi = |z|\cos\theta + i|z|\sin\theta =$$
$$= |z|(\cos\theta + i\sin\theta)$$

Vi definierade förra gången

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

så $z = a+bi = |z|e^{i\theta}$, $\quad \theta = \arg z$

Varför just $e^{i\theta}$?

Ex: $\overline{e^{i\theta}} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) =$
$= e^{i(-\theta)} = e^{-i\theta}$

Ex: $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\underbrace{\cos^2\theta+\sin^2\theta}_{=1}} = \sqrt{1} = 1$

Ex (sats): $\dfrac{1}{e^{i\theta}} = \dfrac{\overline{e^{i\theta}}}{e^{i\theta}\,\overline{e^{i\theta}}} = \dfrac{e^{-i\theta}}{|e^{i\theta}|^2} = \dfrac{e^{-i\theta}}{1^2} = e^{-i\theta}$

Verkar rimligt!

Sats: $e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

②

Bevis: $e^{i\theta_1}\cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$

$$= \underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{=\cos(\theta_1+\theta_2)} + i\underbrace{(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)}_{=\sin(\theta_1+\theta_2)}$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)} \qquad \Box$$

Antag att vi multiplicerar två komplexa tal på polär form: $\quad z_1 = |z_1|e^{i\theta_1}$, $z_2 = |z_2|e^{i\theta_2}$

$$z_1\cdot z_2 = |z_1|e^{i\theta_1}\cdot|z_2|e^{i\theta_2} = (|z_1||z_2|)\cdot e^{i(\theta_1+\theta_2)}$$

> Vid multiplikation multipliceras längder och adderas argument

$$\frac{z_1}{z_2} = \frac{|z_1|e^{i\theta_1}}{|z_2|e^{i\theta_2}} = \left(\frac{|z_1|}{|z_2|}\right)e^{i\theta_1}\cdot\frac{1}{e^{i\theta_2}} =$$
$$= \left(\frac{|z_1|}{|z_2|}\right)\cdot e^{i\theta_1}\cdot e^{-i\theta_2} = \left(\frac{|z_1|}{|z_2|}\right)\cdot e^{i(\theta_1-\theta_2)}$$

> Vid division divideras längder och subtraheras argument

Upprepar vi formeln för multiplikation n ggr. får vi

③

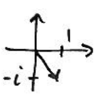
> Sats (De Moivres formel): $(e^{i\theta})^n = e^{in\theta}$, $n \geq 0$
> n heltal

Anm: Fungerar även då $n<0$, ty

$$(e^{i\theta})^{-m} = \frac{1}{(e^{i\theta})^m} = \frac{1}{e^{im\theta}} = e^{-im\theta}, \quad m>0$$

Slutsats: $e^{i\theta}$ följer vanliga räknelagar för potenser!

Ex: $(1-i)^{10} = \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{10} = (\sqrt{2})^{10}e^{-i\frac{\pi}{4}\cdot 10} =$



$$= 2^5 e^{i(-\frac{\pi}{2}-2\pi)} = 32\underbrace{e^{-i\frac{\pi}{2}}}_{=-i}\underbrace{e^{-i2\pi}}_{=1} = \underline{\underline{-32i}}$$

Eulers formler:

$$\begin{cases} ① & e^{i\theta} = \cos\theta + i\sin\theta \\ ② & e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \end{cases}$$

$\dfrac{①+②}{2}$ ger $\quad \boxed{\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}}$

$\dfrac{①-②}{2i}$ ger $\quad \boxed{\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}}$

④

Ex: $\cos^2 x = \left(\dfrac{e^{ix}+e^{-ix}}{2}\right)^2 =$

$$= \frac{1}{4}\Big(e^{i2x} + e^{-i2x} + 2\underbrace{e^{ix}\cdot e^{-ix}}_{=e^0=1}\Big) = \frac{1}{2}\cdot\frac{e^{i2x}+e^{-i2x}}{2} + \frac{1}{2} =$$

$$= \frac{1}{2}\cos 2x + \frac{1}{2} \quad (\text{Jfr. formeln } \cos 2x = 2\cos^2 x - 1)$$

Andragradsekvationer (med komplexa koeff.):

Ex (repetition): $\quad x^2+2x+5 = 0 \iff$

$$x = -1 \pm\sqrt{-4} \iff x = -1 \pm 2i$$

Vi får däremot problem om vi har komplexa koeff., t.ex.

$$z^2 = 1-\sqrt{3}\cdot i \iff z = \pm\sqrt{1-\sqrt{3}\cdot i} \quad ?$$

Vi vet ej vad $\sqrt{1-\sqrt{3}\cdot i}$ betyder! Går ej!

Rätt metod: Sätt $z = a+bi$. Detta ger

$$z^2 = (a+bi)^2 = (a^2-b^2) + 2abi.$$

Vi får $z^2 = 1-\sqrt{3}\cdot i \iff (a^2-b^2)+2abi = 1-\sqrt{3}\cdot i$

Re, Im lika $\iff$
$$\begin{cases} a^2-b^2 = 1 \\ 2ab = -\sqrt{3} \end{cases}$$

Dessutom gäller $|z^2| = |z|^2 = |1-\sqrt{3}\cdot i| = 2$, (5)

och då $|z|^2 = a^2+b^2$ får vi att $a^2+b^2=2$

(hjälpekvation)

Sammanfattningsvis vill vi lösa

$$\begin{cases} a^2-b^2=1 & (1) \\ a^2+b^2=2 & (2) \\ 2ab=-\sqrt{3} & (3) \end{cases} \quad \vdots \quad \begin{matrix} (1)+(2) \\ (2)-(1) \end{matrix} \text{ ger } \begin{cases} 2a^2=3 \\ 2b^2=1 \end{cases}$$

$$\iff \begin{cases} a = \pm\sqrt{\frac{3}{2}} \\ b = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} \end{cases}$$

Ekv. (3) ger att a och b har olika tecken!

Detta ger Svar: $z = a+bi = \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}$ eller

$$z = -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}$$

Alt. lösning (polär form): $z^2 = 1-\sqrt{3}\cdot i$

Skriv om VL och HL på polär form:

$$\begin{cases} z = |z|e^{i\theta} \implies z^2 = |z|^2 e^{i2\theta} \\ 1-\sqrt{3}i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ \qquad = 2e^{-i\frac{\pi}{3}} \end{cases}$$

Detta ger ekv. $|z|^2 e^{i2\theta} = 2e^{-i\frac{\pi}{3}}$

$$\iff \begin{cases} |z|^2 = 2 & \text{(abs.belopp lika)} \\ 2\theta = -\frac{\pi}{3} + 2\pi k & \text{(argument lika)} \end{cases}$$

OBS!

$$\iff \begin{cases} |z| = \underset{(-)}{+}\sqrt{2} \\ \theta = -\frac{\pi}{6} + \pi k, \quad k \text{ heltal,} \end{cases}$$ (6)

vilket ger lösningarna $z_k = \sqrt{2}\, e^{i(-\frac{\pi}{6}+\pi k)}$, k heltal

OBS! Bara $k=0,1$ ger olika lösningar.

Svar: $z_0 = \sqrt{2}e^{-i\frac{\pi}{6}} = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}$$

$$z_1 = \sqrt{2}\, e^{i(-\pi/6+\pi)} = \sqrt{2}\, e^{i\frac{5\pi}{6}} =$$

$$= \sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}$$

Ex: Lös $z^2 - (3-i)z - 10 + \frac{29}{2}i = 0$ (*)

Lösning: $z^2 - (3-i)z - 10 + \frac{29}{2}i = \left(z - \frac{(3-i)}{2}\right)^2 - \frac{(3-i)^2}{4}$

$$-10 + \frac{29}{2}i = \ldots = \left(z - \frac{(3-i)}{2}\right)^2 - 12 + 16i$$

$$(*) \iff \underbrace{\left(z - \frac{(3-i)}{2}\right)}_{=w}{}^2 = 12 - 16i \iff w^2 = 12-16i$$

Vi löser $w^2 = 12-16i$ p.s.s. som ovan (övning!)

och får då $\begin{cases} w_1 = 4-2i \\ w_2 = -4+2i \end{cases}$ vilket ger

Svar: $z_1 = w_1 + \frac{(3-i)}{2} = 4 - 2i + \frac{3}{2} - \frac{i}{2} = \frac{11}{2} - \frac{5}{2}i$ (7)

$$z_2 = w_2 + \frac{(3-i)}{2} = -4 + 2i + \frac{3}{2} - \frac{i}{2} = -\frac{5}{2} + \frac{3}{2}i$$

Binomiska ekvationer (polära metoden):

$$z^n = w \qquad (w \text{ komplext tal})$$

Ex: Lös ekvationen $z^6 = \sqrt{3} + i$!

Lösn.: Sätt $z = |z|e^{i\theta} \implies z^6 = |z|^6 e^{i6\theta}$.

$$\sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2e^{i\frac{\pi}{6}}$$

Detta ger $z^6 = \sqrt{3} + i \iff |z|^6 e^{i6\theta} = 2e^{i\frac{\pi}{6}}$

$$\iff \begin{cases} |z|^6 = 2 \\ 6\theta = \frac{\pi}{6} + 2\pi k \end{cases} \iff \begin{cases} |z| = \underset{(-)}{+} 2^{1/6} & \text{(reellt)} \\ \theta = \frac{\pi}{36} + \frac{\pi}{3}k & \text{(k heltal)} \end{cases}$$

OBS!

så $z_k = 2^{1/6} e^{i(\frac{\pi}{36} + \frac{\pi}{3}k)}$, k heltal

är lösningar.

Vi får bara olika lösningar för $k=0,1,2,3,4$ o 5

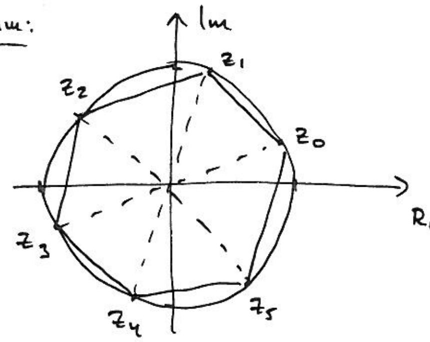
(6 stycken!) eftersom $k=6$ ger vinkeln

$$\frac{\pi}{36} + \frac{\pi}{3}\cdot 6 = \frac{\pi}{36} + 2\pi$$

nytt varv!

Svar: $z_k = 2^{1/6} e^{i(\frac{\pi}{36} + \frac{\pi}{3}k)}$, $k = 0,1,2,3,4,5$. (8)

Anm:



regelbunden sexhörning (hexagon)